INTEGRAL MEANS OF ANALYTIC FUNCTIONS

BY

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ABSTRACT

Sharp bounds for general integral means of analytic functions in the unit disc are determined. These bounds depend only on the moduli of the points on the boundary of the image domain nearest to and farthest from the origin. The proof is shown to be a simple application of a deep theorem of A. Beurling in potential theory.

In this brief note we present a theorem on integral means of analytic functions. Our proof depends entirely on a theorem of A. Beurling [2]. A weaker version would suffice here, but we quote the theorem as in [2]. It has recently come to our attention that a similar theorem is proved by A. Baernstein. Indeed, our theorem follows from a theorem on circular symmetrization and integral means announced in [1].

Let $f(z)$ be univalent in the unit disc, $f(0) = 0$, and let the image domain D of the unit disc under $f(z)$ satisfy the following:

> $r > 0$ is the radius of the largest disc $|w| < r$ contained in D, and R ($0 < r < R$) is the radius of the smallest disc $|w| < R$ containing D.

We denote the class of all such functions by $S(r, R)$. The function $K_{rR}(z) = RK^{-1}[4rRK(z)/(r+R)^2]$, where $K(z) = z/(1+z)^2$, maps the unit disc onto the disc $|w| < R$ slit along the positive real axis from r to R. We denote this domain by Δ.

THEOREM 1. *Let $\phi(t)$ be a convex function of* $\log(t)$ *for* $0 \leqq t \leqq R$ *such that* $\phi(0^+) = \phi(0) < +\infty$ *and* $\phi(R-) \leqq \phi(R) < +\infty$. *Then, for all* $f(z) \in S(r, R)$,

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$$\int_0^{2\pi} \phi(|f(e^{i\theta})|)d\theta \leqq \int_0^{2\pi} \phi(|K_{rR}(e^{i\theta})|)\,d\theta.$$

PROOF. Let $u(w)$ be the harmonic function in D equal to $\phi(|w|)$ on the boundary of D, and let $v(w)$ be the harmonic function in Δ equal to $\phi(|w|)$ on the boundary of Δ. $u(f(z))$ and $v(K_{rR}(z))$ are harmonic in the unit disc $|z| < 1$, and satisfy

$$u(0) = u(f0)) = \frac{1}{2\pi}\int_0^{2\pi} u(f(e^{i\theta}))d0 = \frac{1}{2\pi}\int_0^{2\pi} \phi(|f(e^{i\theta})|)d\theta$$

$$v(0) = v(K_{rR}(0)) = \frac{1}{2\pi}\int_0^{2\pi} v(K_{rR}(e^{i\theta}))d\theta = \frac{1}{2\pi}\int_0^{2\pi} \phi(|K_{rR}(e^{i\theta})|)d\theta.$$

To complete the proof we must show that $u(0) \leqq v(0)$. This follows from Beurling's theorem which we now quote.

DEFINITION ([2, p. 45]). Let D be a domain and R the radius of the largest circle $|w| < R$ containing D. The *projection* E_D is the set of points on the positive real axis which ranges over the moduli of the points w as w ranges over all points in the disc $|w| \leqq R$ *not* contained in D. The domain *associated to* D is the disc $|w| < R$ with the points E_D excluded.

THEOREM 2. ([2, p. 45 and Remark III, p. 52]). *Let D be a bounded domain whose projection is contained in or equal to a finite set of intervals on the positive real axis, (r_k, r_k') $k = 1,\cdots,n$, $0 \leqq r_1 \leqq r_2 \leqq \cdots \leqq r_n' = R$, where R is the radius of the smallest disc $|w| < R$ containing D. Let $\phi(t)$ be a convex function of* $\log(t)$ $0 \leqq t \leqq R$, *such that $\phi(0^+) = \phi(0) < +\infty$, $\phi(R^-) \leqq \phi(R) < +\infty$. Let $v(w)$ be the harmonic function in Δ, the domain associated to D, equal to $\phi(|w|)$ on the boundary of Δ, and let $u(w)$ be harmonic in D, satisfying* $\limsup_{w\to w'} u(w) \leqq \phi(|w'|)$ *at all w' on the boundary of D. Then,* $u(w) \leqq v(-|w|)$.

In our case E_D is the segment $[r, R]$, Δ is the circle $|w| < R$ slit from r to R along the positive real axis, and $u(w)$ satisfies $u(w') = \phi(|w'|)$ for w' on the boundary of Δ. Thus we conclude $u(0) \leqq v(0)$.

Observe that our theorem holds without the condition that $f(z)$ be univalent. Indeed, if $g(z) \in S(r, R)$ but is not univalent, $g(z)$ is subordinate to the univalent function in $S(r, R)$ having the same image, and our result follows from Littlewood's subordination theorem. See [3, p. 10].

According to [1, Th. 2], if $f^*(z)$ is the circular symmetrization of $f(z)$, then (1) holds with $K_{rR}(z)$ replaced by $f^*(z)$. This together with Littlewood's subordination theorem yields our theorem.

We conclude by mentioning that Beurling's theorem provides a proof of the fact that if $f(z) \in S(r, R)$ then $|f'(0)| \leqq 4rR^2/(r+R)^2$. (This is well known, see for example, [4, p. 88].)

Indeed, upon choosing $\phi(t) = \log(t)$, $u(w) - \log(|w|)$ and $v(w) - \log(|w|)$ are the Greens functions of D and Δ respectively with pole at the origin. Now, $|f'(0)| = \exp(u(0))$ and $|K'_{rR}(0)| = \exp(v(0)) = 4rR^2/(r+R)^2$, and the result follows.

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References

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